

# ON THE STABILITY OF THE ERDŐS-KO-RADO THEOREM

BÉLA BOLLOBÁS, BHARGAV P. NARAYANAN, AND ANDREI M. RAIGORODSKII

ABSTRACT. Delete the edges of a Kneser graph with some probability  $p$ , independently of each other: for what probabilities  $p$  is the independence number of this random graph equal to the independence number of the Kneser graph itself? We prove a sharp threshold result for this question in certain regimes. Since an independent set in the Kneser graph is the same as an intersecting (uniform) family, this gives us a random analogue of the Erdős-Ko-Rado theorem.

## 1. INTRODUCTION

In this note, our aim is to investigate the stability of a central result in extremal set theory due to Erdős, Ko and Rado [12]. A family of sets  $\mathcal{A}$  is said to be *intersecting* if  $A \cap B \neq \emptyset$  for all  $A, B \in \mathcal{A}$ . We are interested in intersecting families where all the sets have the same size; writing  $[n]$  for the set  $\{1, 2, \dots, n\}$  and  $[n]^{(r)}$  for the family of all the subsets of  $[n]$  of cardinality  $r$ , the Erdős-Ko-Rado theorem asserts that if  $\mathcal{A} \subset [n]^{(r)}$  is intersecting and  $n \geq 2r$ , then  $|\mathcal{A}| \leq \binom{n-1}{r-1}$ . It is easy to see that equality is achieved when  $\mathcal{A}$  is the family of all the  $r$ -element subsets of  $[n]$  containing a fixed element  $x \in [n]$ ; we call this family the *star centred at  $x$* . Hilton and Milnor [16] showed, in fact, that if  $n > 2r$ , then the *only* intersecting families of size  $\binom{n-1}{r-1}$  are the trivial ones, namely, stars. Many extensions of the Erdős-Ko-Rado theorem and the Hilton-Milnor theorem have since been proved; furthermore, very general stability results about the structure of intersecting families have been proved by Friedgut [13], Dinur and Friedgut [11], and Keevash and Mubayi [17].

Here, we shall investigate a different notion of stability and prove a random analogue of the Erdős-Ko-Rado theorem which strengthens the Erdős-Ko-Rado theorem significantly when  $r$  is small compared to  $n$ .

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To translate the Erdős-Ko-Rado theorem to the random setting, it will be helpful to reformulate the theorem as a statement about Kneser graphs. For natural numbers  $n, r \in \mathbb{N}$  with  $n \geq r$ , the Kneser graph  $K(n, r)$  is the graph whose vertex set is  $[n]^{(r)}$  where two  $r$ -element sets  $A, B \in [n]^{(r)}$  are adjacent if and only if  $A \cap B = \emptyset$ . Observe that a family  $\mathcal{A} \subset [n]^{(r)}$  is an intersecting family if and only if  $\mathcal{A}$  is an independent set in  $K(n, r)$ . Writing  $\alpha(G)$  for the size of the largest independent set in a graph  $G$ , the Erdős-Ko-Rado theorem asserts that  $\alpha(K(n, r)) = \binom{n-1}{r-1}$  when  $n \geq 2r$ ; furthermore, when  $n > 2r$ , the only independent sets of this size are stars.

Let us now delete the edges of the Kneser graph  $K(n, r)$  with some probability  $p$ , independently of each other. When is the independence number of this random subgraph equal to  $\binom{n-1}{r-1}$ ? It turns out that when  $r$  is much smaller than  $n$ , an analogue of the Erdős-Ko-Rado theorem continues to be true even after we delete practically all the edges of the Kneser graph!

This kind of phenomenon, namely the validity of classical extremal results for surprisingly sparse random structures, has received a lot of attention over the past twenty five years.

Perhaps the first result of this kind in extremal graph theory was proved by Babai, Simonovits, and Spencer [1] who showed that an analogue of Mantel's theorem is true for certain random graphs. Mantel's theorem states that the largest triangle free subgraph and the largest bipartite subgraph of  $K_n$ , the complete graph on  $n$  vertices, have the same size. Babai, Simonovits, and Spencer proved that the same holds for the Erdős-Rényi random graph  $G(n, p)$  with high probability when  $p \geq 1/2 - \delta$  for some absolute constant  $\delta > 0$ . In other words, they show that Mantel's theorem is 'stable' in the sense that it holds not only for the complete graph but that it holds *exactly* for random subgraphs of the complete graph as well. Improving upon results of Brightwell, Panagiotou and Steger [6], DeMarco and Kahn [9] have recently shown that this phenomenon continues to hold even when the random graph  $G(n, p)$  is very sparse; they show in particular that it suffices to take  $p \geq C(\log n/n)^{1/2}$  for some absolute constant  $C > 0$ , and that this is best possible up to the value of the absolute constant.

The first such transference results in Ramsey theory were proved by Rödl and Ruciński [21, 22] and there have been many related Ramsey theoretic results since; see, for example, [14, 23, 19].

Phenomena of this kind have also been observed in additive combinatorics. Roth's theorem [25], a central result in additive combinatorics, states that for every  $\delta > 0$  and all sufficiently large  $n$ , every subset of  $[n] = \{1, 2, \dots, n\}$  of density  $\delta$  contains a three-term arithmetic progression. Kohayakawa, Rödl and Łuczak [18] proved a random analogue, showing that such a statement holds not only for  $[n]$  but also, with high probability, for random subsets of  $[n]$  of density at least  $Cn^{-1/2}$ , where  $C > 0$  is an absolute constant.

Another classical result in additive combinatorics, due to Diananda and Yap [10], is that the largest sum-free subset of  $\mathbb{Z}_{2n}$  is the set of odd numbers. Balogh, Morris and Samotij [3] proved that the same is true of random subsets of  $\mathbb{Z}_{2n}$  of density greater than  $(\log n/3n)^{1/2}$  with high probability, and also that this no longer the case when the density is less than  $(\log n/3n)^{1/2}$ . Thus, there is a *sharp threshold* at  $(\log n/3n)^{1/2}$  for the stability of this extremal result; an extension of this sharp threshold result to all even-order Abelian groups has recently been proved by Bushaw, Neto, Morris and Smith [7].

Perhaps the most striking application of such transference principles in additive combinatorics is the Green-Tao theorem [15] on primes in arithmetic progressions.

These results constitute a tiny sample of the large number of beautiful results which have been proved in this setting. Very general transference theorems have been proved by Conlon and Gowers [8] and Schacht [26]. We refer the interested reader to the surveys of Łuczak [20] and Rödl and Schacht [24] for a more detailed account of such results.

Returning to the question at hand, our aim, as we remarked before, is to investigate the independence number of random subgraphs of  $K(n, r)$ . (The independence number of random *induced* subgraphs of  $K(n, r)$ , i.e., the size of the largest intersecting subfamily of a random  $r$ -uniform hypergraph, has been investigated by Balogh, Bohman and Mubayi [2].) As we mentioned earlier, an analogue of the Erdős-Ko-Rado theorem continues to hold for very sparse random subgraphs of the  $K(n, r)$  and so it will be more convenient to retain (rather than delete) the edges of  $K(n, r)$  independently with some probability  $p$ ; to this end, let  $K_p(n, r)$  denote the random subgraph of  $K(n, r)$  obtained by retaining each edge of  $K(n, r)$  independently with probability  $p$ . The main question of interest is the following.

**Problem 1.1.** *For what  $p > 0$  is  $\alpha(K_p(n, r)) = \binom{n-1}{r-1}$  with high probability?*

For constant  $r$  and  $n$  sufficiently large, a partial answer was provided by Bogolyubskiy, Gusev, Pyaderkin and Raigorodskii [5, 4]: they studied random subgraphs of  $K(n, r, s)$ , where  $K(n, r, s)$  is the graph whose vertex set is  $[n]^{(r)}$  where two  $r$ -element sets  $A, B \in [n]^{(r)}$  are adjacent if and only if  $|A \cap B| = s$ ; in the case  $s = 0$  (which corresponds to the Kneser graph), they established that  $\alpha(K_{1/2}(n, r)) = (1 + o(1)) \binom{n-1}{r-1}$  with high probability.

We shall do much more and answer Question 1.1 exactly when  $r$  is small compared to  $n$  (more precisely, when  $r = o(n^{1/3})$ ). To state our result, it will be convenient to define the threshold function

$$p_c(n, r) = \frac{(r+1) \log n - r \log r}{\binom{n-1}{r-1}}. \quad (1)$$

With this definition in place, we can now state our main result.

**Theorem 1.2.** *Fix a real number  $\varepsilon > 0$  and let  $r = r(n)$  be a natural number such that  $2 \leq r(n) = o(n^{1/3})$ . Then as  $n \rightarrow \infty$ ,*

$$\mathbb{P}\left(\alpha(K_p(n, r)) = \binom{n-1}{r-1}\right) \rightarrow \begin{cases} 1 & \text{if } p \geq (1 + \varepsilon)p_c(n, r) \\ 0 & \text{if } p \leq (1 - \varepsilon)p_c(n, r). \end{cases}$$

*Furthermore, when  $p \geq (1 + \varepsilon)p_c$ , the only independent sets of size  $\binom{n-1}{r-1}$  in  $K_p(n, r)$  are the trivial ones, namely, stars.*

This paper is organised as follows. We establish some notation and collect together some standard facts in Section 2. Most of the work involved in proving Theorem 1.2 is in establishing the upper bound on the critical density; we do this in Section 3. We complete the proof of Theorem 1.2 by proving a matching lower bound in Section 4. We conclude with some discussion in Section 5.

## 2. PRELIMINARIES

**2.1. Notation.** Given  $x \in [n]$  and  $\mathcal{A} \subset [n]^{(r)}$ , we write  $\mathcal{S}_x$  for the star centred at  $x$ , and  $\mathcal{A}_x$  for the subfamily of  $\mathcal{A}$  consisting of those sets (of  $\mathcal{A}$ ) that contain  $x$ , i.e.,  $\mathcal{A}_x = \mathcal{A} \cap \mathcal{S}_x$ . The maximum degree  $d(\mathcal{A})$  of a family  $\mathcal{A} \subset [n]^{(r)}$  is defined to be the maximum cardinality, over all  $x \in [n]$ , of a subfamily  $\mathcal{A}_x$ , and we write  $e(\mathcal{A})$  for the number of edges in  $K(n, r)$  induced by  $\mathcal{A}$ . Since any pair of intersecting sets  $A, B \in \mathcal{A}$  both belong to at least one subfamily  $\mathcal{A}_x$ , we get the following estimate for  $e(\mathcal{A})$  which is useful when the maximum degree of  $\mathcal{A}$  is small.

**Proposition 2.1.** *For any  $\mathcal{A} \subset [n]^{(r)}$ ,*

$$e(\mathcal{A}) \geq \binom{|\mathcal{A}|}{2} - \sum_{x \in [n]} \binom{|\mathcal{A}_x|}{2}. \quad \square$$

To ease the notational burden, in the rest of this paper, we shall write  $R = \binom{n}{r}$  for the size of  $[n]^{(r)}$ , and  $N = \binom{n-1}{r-1}$  for the size of a star. Also, given  $x \in [n]$  and a set  $A \in [n]^{(r)}$  not containing  $x$ , we shall write  $M = \binom{n-r-1}{r-1}$  for the number of sets of  $\mathcal{S}_x$  disjoint from  $A$ .

A word on asymptotic notation; we use the standard  $o(1)$  notation to denote any function that tends to zero as  $n$  tends to infinity. Here and elsewhere, the variable tending to infinity will always be  $n$  unless we explicitly specify otherwise.

**2.2. Estimates.** Next, we collect some standard estimates that we shall use repeatedly; for ease of reference, we list them as propositions below.

Let us start with a weak form of Sterling's approximation for the factorial function.

**Proposition 2.2.** *For all  $n \in \mathbb{N}$ ,*

$$\sqrt{2\pi n} \left(\frac{n}{e}\right)^n \leq n! \leq e^{1/12n} \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \quad \square$$

In fact, the following crude bounds for the binomial coefficients will often be sufficient for our purposes.

**Proposition 2.3.** *For all  $n, r \in \mathbb{N}$ ,*

$$\left(\frac{n}{r}\right)^r \leq \binom{n}{r} \leq \frac{n^r}{r!} \leq \left(\frac{en}{r}\right)^r. \quad \square$$

Also, we will need the following standard inequality concerning the exponential function.

**Proposition 2.4.** *For every  $x \in \mathbb{R}$  such that  $|x| \leq 1/2$ ,*

$$e^{x-x^2} < 1+x < e^x. \quad \square$$

Although our last proposition is also very simple, we prove it here for the sake of completeness. Recall that  $N = \binom{n-1}{r-1}$  and  $M = \binom{n-r-1}{r-1}$ .

**Proposition 2.5.** *If  $r = r(n) = o(n^{1/2})$ , then  $N - M = o(N)$ . Furthermore, if  $r = o(n^{1/3})$ , then  $N - M = o(N/r)$ .*

*Proof.* Both claims follow from the observation that

$$\begin{aligned}
N - M &= \binom{n-1}{r-1} - \binom{n-r-1}{r-1} \\
&= \sum_{i=1}^r \binom{n-i}{r-1} - \binom{n-i-1}{r-1} \\
&= \sum_{i=1}^r \binom{n-i-1}{r-2} \\
&\leq r \binom{n-2}{r-2} = \frac{r(r-1)}{n-1} N. \quad \square
\end{aligned}$$

### 3. UPPER BOUND FOR THE CRITICAL THRESHOLD

We now turn to our proof of Theorem 1.2. In this section, we shall bound the critical threshold from above, i.e., we shall prove that a random analogue of the Erdős-Ko-Rado theorem holds if  $p > (1 + \varepsilon)p_c(n, r)$  where  $p_c(n, r)$  is given by (1).

*Proof of the upper bound in Theorem 1.2.* Let  $0 < \varepsilon < 1/2$  and set  $p = p(n) = (1 + \varepsilon)p_c(n, r)$ . We shall prove that with high probability, the independence number of  $K_p(n, r)$  is  $N$ , and that furthermore, the only independent sets of size  $N$  in  $K_p(n, r)$  are stars. Since we are working with monotone properties, it suffices to prove this result for  $\varepsilon$  small enough and so we lose nothing by assuming  $0 < \varepsilon < 1/2$ .

For each  $i \geq 1$ , let  $X_i$  be the number of families  $\mathcal{A} \subset [n]^{(r)}$  inducing an independent set in  $K_p(n, r)$  such that  $|\mathcal{A}| = N$  and  $d(\mathcal{A}) = N - i$ . Also, let  $Y$  be the number of independent families  $\mathcal{A} \subset [n]^{(r)}$  such that  $|\mathcal{A}| = N + 1$  and  $d(\mathcal{A}) = N$ ; in other words, independent families of size  $N + 1$  which contain an entire star.

Our aim is to show that with high probability, the random variables defined above are all equal to zero. This then implies the lower bound on the critical threshold; since every  $X_i$  is equal to zero, every independent set in  $K_p(n, r)$  of cardinality at least  $N$  must contain an entire star, and since  $Y$  is also equal to zero, the only independent sets of cardinality at least  $N$  are stars.

We start by computing  $\mathbb{E}[Y]$ . We know that for any star  $\mathcal{S}$ , any  $A \in [n]^{(r)} \setminus \mathcal{S}$  is disjoint from  $M$  elements of  $\mathcal{S}$ , and so,

$$\mathbb{E}[Y] = \binom{n}{1} \binom{R-N}{1} (1-p)^M. \quad (2)$$

When  $r = o(n^{1/3})$  (indeed, when  $r = o(n^{1/2})$ ), we know from Proposition 2.5 that  $M = (1 + o(1))N$ . Since  $p = (1 + \varepsilon)((r + 1) \log n - r \log r)/N$ , we see that

$$\begin{aligned} \mathbb{E}[Y] &\leq nR(1 - p)^{(1+o(1))N} \\ &\leq n\left(\frac{en}{r}\right)^r \exp((-1 + o(1))pN) \\ &\leq n\left(\frac{en}{r}\right)^r \exp((1 + \varepsilon + o(1))(r \log r - (r + 1) \log n)) \\ &\leq \left(\frac{r}{n}\right)^{(\varepsilon+o(1))r} \leq n^{-(\varepsilon+o(1))4r/5} = o(1). \end{aligned}$$

By Markov's inequality, we know that  $\mathbb{P}(Y > 0) \leq \mathbb{E}[Y]$  and it follows that  $Y$  is zero with high probability.

We now turn our attention to the  $X_i$ . To keep our argument simple, we distinguish three cases: we first deal with small values of  $i$  where the  $X_i$  count families of very large maximum degree, then we consider families of large (but not huge) maximum degree, and in the final case, we deal with families of small maximum degree.

**Case 1: Very large maximum degree.** Unfortunately, when  $i$  is small, it is not true that  $\mathbb{E}[X_i]$  goes to zero as  $n$  grows. For constant  $i$ ,  $\mathbb{E}[X_i] \geq n \binom{N}{i} \binom{R}{i} (1 - p)^{(i+o(1))N}$ . When  $r = 3$  and  $i = 2$  for example, it follows that

$$\begin{aligned} \mathbb{E}[X_2] &\geq n \binom{\binom{n-1}{2}}{2} \binom{\binom{n}{3}}{2} (1 - p)^{(2+o(1))N} \\ &\geq n^{o(1)} \frac{n^{11}}{n^{8(1+\varepsilon)}} \geq n^{3-8\varepsilon+o(1)}, \end{aligned}$$

which grows with  $n$  when  $\varepsilon$  is small enough. However, if we compute  $\text{Var}[X_2]$ , we are encouraged to find that  $\text{Var}[X_2]/\mathbb{E}[X_2]^2$  is bounded away from one; indeed, we observe similar behaviour for any fixed value of  $i$  and larger  $r$  as well. We therefore adopt a different strategy to bound  $\mathbb{P}(X_i > 0)$  for small  $i$ .

Note that any family counted by  $X_i$  can be described by specifying a star  $\mathcal{S}$ , a subfamily  $\mathcal{A}_1 \subset \mathcal{S}$  of  $i$  sets missing from  $\mathcal{S}$ , and another family  $\mathcal{A}_2$  of cardinality  $i$  disjoint from  $\mathcal{S}$  such that all the edges between  $\mathcal{S} \setminus \mathcal{A}_1$  and  $\mathcal{A}_2$  in  $K(n, r)$  are absent in  $K_p(n, r)$ . (Of course, it is also true that the edges induced by  $\mathcal{A}_2$  must also be absent in  $K_p(n, r)$ , but we won't need this fact.) Consequently, such a configuration gives rise to a family counted by  $Y$  if there exists a set  $A \in \mathcal{A}_2$  such that all the edges between  $A$  and  $\mathcal{A}_1$  in  $K(n, r)$  (of which there are at most  $i$ ) are

absent in  $K_p(n, r)$ . This implies that

$$\mathbb{P}(Y > 0 \mid X_i > 0) \geq i(1-p)^i$$

and so

$$\mathbb{P}(X_i > 0) \leq \mathbb{E}[Y]/i(1-p)^i.$$

Summing this estimate for  $i \leq \varepsilon N/2$ , we get

$$\begin{aligned} \sum_{i=1}^{\varepsilon N/2} \mathbb{P}(X_i > 0) &\leq \sum_{i=1}^{\varepsilon N/2} \frac{\mathbb{E}[Y]}{i(1-p)^i} \\ &\leq \mathbb{E}[Y](1-p)^{-\varepsilon N/2} \log N \\ &\leq nR(1-p)^{(1-\varepsilon/2+o(1))N} \log N \\ &\leq n\left(\frac{en}{r}\right)^r \exp\left(\left(1 + \frac{\varepsilon}{2} - \frac{\varepsilon^2}{2} + o(1)\right)p_c(n, r)N\right) \log N \\ &\leq r \log n \left(\frac{r}{n}\right)^{(\varepsilon/2 - \varepsilon^2/2 + o(1))r} = o(1), \end{aligned}$$

and so by the union bound, with high probability, for each  $1 \leq i \leq \varepsilon N/2$ , the random variable  $X_i$  is zero.

**Case 2: Large maximum degree.** Next, we consider the  $X_i$  with

$$\varepsilon N/2 < i \leq N\left(1 - \frac{1 - \varepsilon/2}{r+1}\right).$$

For any star  $\mathcal{S}$ , we know that the number of edges in  $K(n, r)$  between a set  $A \in [n]^{(r)} \setminus \mathcal{S}$  and a family  $\mathcal{A} \subset \mathcal{S}$  is at least  $|\mathcal{A}| - (N - M)$ . We know from Proposition 2.5 that  $N - M = o(N/r)$  when  $r = o(n^{1/3})$ ; consequently, it follows that if  $\mathcal{A} \subset [n]^{(r)}$  has cardinality  $N$  and  $d(\mathcal{A}) \geq (1 - \varepsilon/2)N/(r+1)$ , then  $e(\mathcal{A}) \geq (1 + o(1))d(\mathcal{A})(N - d(\mathcal{A}))$ .

To simplify calculations, let us set  $i = \alpha N = \alpha r R/n$  where

$$\varepsilon/2 < \alpha \leq (r + \varepsilon/2)/(r+1).$$

In this range, we see that

$$\begin{aligned} \mathbb{E}[X_i] &\leq n \binom{N}{i} \binom{R}{i} (1-p)^{e(\mathcal{A})} \\ &\leq n \left(\frac{e}{\alpha}\right)^{\alpha N} \left(\frac{en}{r\alpha}\right)^{\alpha N} \exp\left((1 + \varepsilon + o(1))\alpha(1 - \alpha)p_c(n, r)N^2\right) \end{aligned}$$



$$\leq n \left( \frac{n r^{(1+\varepsilon+o(1))(1-\alpha)r}}{r n^{(1+\varepsilon+o(1))(1-\alpha)(r+1)}} \right)^{\alpha N} \leq n \left( \frac{r}{n} \right)^{(\varepsilon^2/4 - \varepsilon^3/4 + o(1))N},$$

and it follows that

$$\sum_{i=\varepsilon N/2}^{\frac{(2r+\varepsilon)N}{2(r+1)}} \mathbb{P}(X_i > 0) \leq n N \left( \frac{r}{n} \right)^{(\varepsilon^2/4 - \varepsilon^3/4 + o(1))N} = o(1),$$

and so with high probability, for each  $\varepsilon N/2 < i \leq (r + \varepsilon/2)N/(r+1)$ , the random variable  $X_i$  is zero.

**Case 3: Small maximum degree.** We shall complete the proof of the lower bound by showing that

$$\sum_{i > \frac{(2r+\varepsilon)N}{2(r+1)}} \mathbb{E}[X_i] = o(1).$$

It turns out that in this range of  $i$ , somewhat surprisingly, it is significantly easier to deal with the case where  $r$  tends to infinity with  $n$  as opposed to the case where  $r$  is small.

Suppose first that  $r \geq \log n$ . Then

$$\frac{(1 - \varepsilon/2)}{r + 1} < \frac{1}{r + 4}$$

for all large enough  $n$ . Observe that subgraph of  $K(n, r)$  induced by  $\mathcal{A}$  has minimum degree at least  $N - rd(\mathcal{A})$  and consequently, if  $d(\mathcal{A}) < N/(r + 4)$ , then

$$e(\mathcal{A}) \geq \frac{N}{2} \left( N - \frac{rN}{r + 4} \right) = \frac{2N^2}{r + 4}.$$

In this case, it follows that

$$\begin{aligned} \sum_{i > \frac{(2r+\varepsilon)N}{2(r+1)}} \mathbb{E}[X_i] &\leq \binom{R}{N} (1 - p)^{2N^2/(r+4)} \\ &\leq \left( \frac{en}{r} \right)^N \left( \frac{r}{n} \right)^{2rN/(r+4)} \\ &\leq \left( \frac{er}{n} \right)^{(1+o(1))N} = o(1) \end{aligned}$$

which completes the proof when  $r \geq \log n$ .

Next, suppose that  $r \leq \log n$ . When  $r \leq \log n$ , it is not necessarily true (if  $r = O(1)$  and  $\varepsilon$  is sufficiently small, for instance) that  $(1 - \varepsilon/2)/(r + 1) < 1/(r + 4)$ . It turns out that in this case, we need a more careful estimate.

For a family  $\mathcal{A} \subset [n]^{(r)}$  and each  $x \in [n]$ , define  $\alpha_x = |\mathcal{A}_x|/N$ . Note that  $\sum_{x=1}^n \alpha_x = r$ . Recall that Proposition 2.1 tells us that

$$e(\mathcal{A}) \geq \binom{|\mathcal{A}|}{2} - \sum_{x \in [n]} \binom{|\mathcal{A}_x|}{2} \geq \binom{N}{2} \left( 1 - \sum_{x \in [n]} \alpha_x^2 \right).$$

Let  $\mathcal{A} \subset [n]^{(r)}$  be such that  $d(\mathcal{A}) < (1 - \varepsilon/2)N/(r+1)$ . For such a family  $\mathcal{A}$ , let  $D = D_{\mathcal{A}}$  be the set of  $x \in [n]$  such that  $\alpha_x \geq (\log n)^{-2}$ . Since  $\sum_{x=1}^n \alpha_x = r$ , we see that  $|D| \leq r(\log n)^2 \leq (\log n)^3$ .

**Lemma 3.1.** *Fix  $D = D_{\mathcal{A}}$  and the values of  $|\mathcal{A}_x|$  for  $x \in D$ . Subject to these restrictions, the expected number of families  $\mathcal{A} \subset [n]^{(r)}$  of maximum degree at most  $(1 - \varepsilon/2)N/(r+1)$  which induce independent sets in  $K_p(n, r)$  is at most  $(r/n)^{(3/10+o(1))N}$ .*

*Proof.* Since  $\sum_{x=1}^n \alpha_x = r$  and  $|D| \leq (\log n)^3$ , it follows by convexity that  $\sum_{x \in [n] \setminus D} \alpha_x^2$  is at most  $|D|/(\log n)^4 \leq (\log n)^{-1} = o(1)$ . Consequently,

$$e(\mathcal{A}) \geq \frac{N^2}{2} \left( 1 - \sum_{x \in [n]} \alpha_x^2 \right) \geq \frac{N^2}{2} \left( 1 + o(1) - \sum_{x \in D} \alpha_x^2 \right),$$

and so the probability that a family  $\mathcal{A}$  as in the statement of the lemma induces an independent set is at most

$$\begin{aligned} (1-p)^{e(\mathcal{A})} &\leq \exp \left( -\frac{pN^2}{2} \left( 1 + o(1) - \sum_{x \in D} \alpha_x^2 \right) \right) \\ &\leq \left( \frac{r^{(1+o(1))r}}{n^{(1+o(1))(r+1)}} \prod_{x \in D} \left( \frac{n^{r+1}}{r^r} \right)^{\alpha_x^2} \right)^{N/2}. \end{aligned} \quad (3)$$

Next, we bound the number of ways in which we can choose  $\mathcal{A}$  as in Lemma 3.1. Using the fact that  $r \leq \log n$  and  $|D| \leq (\log n)^3$ , we first note that

$$\begin{aligned} N &\geq \left| \bigcup_{x \in D} \mathcal{A}_x \right| \geq \sum_{x \in D} |\mathcal{A}_x| - \sum_{\substack{x, y \in D \\ x < y}} |\mathcal{A}_x \cap \mathcal{A}_y| \\ &\geq \sum_{x \in D} |\mathcal{A}_x| - |D|^2 \binom{n-2}{r-2} \\ &\geq \left( \sum_{x \in D} \alpha_x - \frac{|D|^2 r}{n} \right) N \geq \left( \sum_{x \in D} \alpha_x + o(1) \right) N. \end{aligned}$$

It follows that

$$\sum_{x \in D} \alpha_x \leq 1 + o(1) < 1 + 1/10 \quad (4)$$

and

$$|\mathcal{A} \setminus (\cup_{x \in D} \mathcal{A}_x)| < N \left( 1 + 1/5 - \sum_{x \in D} \alpha_x \right).$$

(Here, the choice of the constants  $1/10$  and  $1/5$  was arbitrary; any two sufficiently small constants would have sufficed.) Hence, the number of ways to choose  $\mathcal{A}$  is at most

$$\begin{aligned} \binom{R}{N(6/5 - \sum_{x \in D} \alpha_x)} \prod_{x \in D} \binom{N}{\alpha_x N} &\leq \left( \frac{10en}{r} \right)^{N(6/5 - \sum_{x \in D} \alpha_x)} \prod_{x \in D} \left( \frac{e}{\alpha_x} \right)^{\alpha_x N} \\ &\leq 100^N \left( \frac{n}{r} \right)^{6N/5} \prod_{x \in D} \left( \frac{r}{\alpha_x n} \right)^{\alpha_x N} \\ &\leq \left( \frac{n}{r} \right)^{(6/5 + o(1))N} \prod_{x \in D} \left( \frac{r}{\alpha_x n} \right)^{\alpha_x N}. \end{aligned} \quad (5)$$

From (3) and (5), we conclude that the expected number of independent families  $\mathcal{A}$  as in the lemma is at most

$$\begin{aligned} \left( \frac{r^{(1+o(1))r/2-6/5}}{n^{(1+o(1))(r+1)/2-6/5}} \right)^N \prod_{x \in D} \left( \left( \frac{r}{\alpha_x n} \right) \left( \frac{n^{r+1}}{r^r} \right)^{\alpha_x/2} \right)^{\alpha_x N} \\ \leq \left( \frac{r^{(1+o(1))r/2-6/5}}{n^{(1+o(1))(r+1)/2-6/5}} \right)^N. \end{aligned} \quad (6)$$

In the inequality above, we used the fact that  $(r/\alpha n)(n^{r+1}/r^r)^{\alpha/2} < 1$  whenever  $(\log n)^{-2} \leq \alpha < (1 - \varepsilon/2)/(r+1)$ . To see this, observe that the function  $f$  defined on the positive reals by

$$f(\alpha) = \frac{\alpha((r+1)\log n - r\log r)}{2} - \log \alpha + \log(r/n)$$

has exactly one extremum at  $\alpha_0 = 2/((r+1)\log n - r\log r)$  and is increasing when  $\alpha > \alpha_0$  and decreasing when  $\alpha < \alpha_0$ ; so to check that  $f(\alpha) < 0$  when  $(\log n)^{-2} \leq \alpha \leq (1 - \varepsilon/2)/(r+1)$ , it suffices to check that  $f((\log n)^{-2}) < 0$  and  $f((1 - \varepsilon/2)/(r+1)) < 0$ . Since both conditions hold for all sufficiently large  $n$  when  $r \leq \log n$ , we see that (6) holds.

Since  $(r+1)/2 - 6/5 \geq 3/10$  for all  $r \geq 2$ , we conclude that the right-hand side of (6) is at most

$$\left( \frac{r^{(1+o(1))r/2-6/5}}{n^{(1+o(1))(r+1)/2-6/5}} \right)^N \leq \left( \frac{r^{(1+o(1))(r+1)/2-6/5}}{n^{(1+o(1))(r+1)/2-6/5}} \right)^N \leq \left( \frac{r}{n} \right)^{(3/10+o(1))N}.$$

This completes the proof of Lemma 3.1.  $\square$

Recall that if  $r \leq \log n$  and  $d(\mathcal{A}) < (1 - \varepsilon/2)N/(r+1)$ , then  $|D_{\mathcal{A}}| \leq (\log n)^3$ . So the number of choices for the set  $D_{\mathcal{A}}$  is clearly at most

$$\sum_{j=0}^{(\log n)^3} \binom{n}{j} \leq (\log n)^3 \binom{n}{(\log n)^3}. \quad (7)$$

We know from (4) that the values  $|\mathcal{A}_x|$  for  $x \in D_{\mathcal{A}}$  satisfy

$$\sum_{x \in D} |\mathcal{A}_x| \leq 11N/10$$

and so, the number of ways of selecting the values of  $|\mathcal{A}_x|$  is at most

$$\binom{11N/10 + (\log n)^3 + 1}{(\log n)^3} \leq (2N)^{(\log n)^3}. \quad (8)$$

From Lemma 3.1, we conclude using (7) and (8) that

$$\sum_{i > \frac{(2r+\varepsilon)N}{2(r+1)}} \mathbb{E}[X_i] \leq (\log n)^3 n^{(\log n)^3} (2N)^{(\log n)^3} \left( \frac{r}{n} \right)^{(3/10+o(1))N}. \quad (9)$$

It is easy to check that the right-hand side of (9) is  $o(1)$  for every  $2 \leq r \leq \log n$ . Hence, with high probability, for each  $i > (r + \varepsilon/2)N/(r+1)$ , the random variable  $X_i$  is zero; this completes the proof of the lower bound.  $\square$

*Remark.* A more careful analysis can be used to show that for large  $r$ , i.e., when  $r$  tends to infinity with  $n$ , it is sufficient to take  $\varepsilon$  to be greater than  $6/r$ , say, as opposed to a small fixed constant.

#### 4. LOWER BOUND FOR THE CRITICAL THRESHOLD

As in the previous section, let  $Y$  be the number of independent families in  $K_p(n, r)$  of size  $N+1$  which contain an entire star.

*Proof of the lower bound in Theorem 1.2.* Turning to the lower bound, we will assume that  $p = (1 - \varepsilon)p_c(n, r)$  for some fixed real number  $\varepsilon > 0$  and we show using a simple second moment calculation that  $Y > 0$  with high probability; consequently, the independence number of  $K_p(n, r)$  is at least  $N + 1$ .

Recall (2) which says that

$$\mathbb{E}[Y] = \binom{n}{1} \binom{R - N}{1} (1 - p)^M.$$

Note that  $N = o(R)$  when  $r = o(n^{1/3})$ ; it follows that

$$\begin{aligned} \mathbb{E}[Y] &\geq (1 + o(1))nR(1 - p)^N \\ &\geq (1 + o(1))\frac{n^{r+1}}{r!} \exp(-(p + p^2)N) \\ &\geq \frac{n^{r+1}}{r!} \exp((1 - \varepsilon + o(1))(r \log r - (r + 1) \log n)) \\ &\geq \left(\frac{n}{r}\right)^{(\varepsilon + o(1))r}, \end{aligned}$$

and so  $\mathbb{E}[Y] \rightarrow \infty$  when  $p = (1 - \varepsilon)p_c(n, r)$ .

Therefore, to show that  $Y > 0$  with high probability, it suffices to show that  $\text{Var}[Y] = o(\mathbb{E}[Y]^2)$  or equivalently, that  $\mathbb{E}[(Y)_2] = (1 + o(1))\mathbb{E}[Y]^2$ , where  $\mathbb{E}[(Y)_2] = \mathbb{E}[Y(Y - 1)]$  is the second factorial moment of  $Y$ .

Note that

$$\mathbb{E}[(Y)_2] = \sum_{x, y, A, B} \mathbb{P}(\mathcal{S}_x \cup \{A\} \text{ and } \mathcal{S}_y \cup \{B\} \text{ are independent}),$$

the sum being over ordered four tuples  $(x, y, A, B)$  with  $x, y \in [n]$ ,  $A \in [n]^{(r)} \setminus \mathcal{S}_x$  and  $B \in [n]^{(r)} \setminus \mathcal{S}_y$  such that  $(x, A) \neq (y, B)$ . Now, observe that

$$\begin{aligned} \sum_{x \neq y} \mathbb{P}(\mathcal{S}_x \cup \{A\} \text{ and } \mathcal{S}_y \cup \{B\} \text{ are independent}) &= (n^2 - n)(R - N)^2(1 - p)^{2M} \\ &= (1 + o(1))\mathbb{E}[Y]^2, \end{aligned}$$

and

$$\begin{aligned} \sum_{x=y, A \neq B} \mathbb{P}(\mathcal{S}_x \cup \{A\} \text{ and } \mathcal{S}_y \cup \{B\} \text{ are independent}) &\leq n(R - N)^2(1 - p)^{2M} \\ &= o(\mathbb{E}[Y]^2). \end{aligned}$$

By Chebyshev's inequality, we conclude that  $Y > 0$  with high probability and so, the independence number of  $K_p(n, r)$  is at least  $N + 1$ .  $\square$

## 5. CONCLUDING REMARKS

The condition  $r = o(n^{1/3})$  in our results seems somewhat artificial; we would expect the same formula for the critical threshold to hold for much larger  $r$  as well. We suspect that this formula in fact gives the exact value of the critical threshold when  $r = o(n^{1/2})$ , and potentially when  $r = O(n^{1-\delta})$  for any absolute constant  $\delta > 0$ , but we are unable to prove this presently.

The size of the critical window also merits study. As we remarked earlier, our proof (for large  $r$ ) works even when we are a factor of  $(1 + 6/r)$  away from the critical threshold; it is likely that the critical window is much smaller.

Of course, one would be interested to know what happens for larger  $r$  as well. When  $r/n$  is bounded away from  $1/2$ , we suspect it should be possible to demonstrate stability of the Erdős-Ko-Rado theorem at  $p = 1/2$ , say. Perhaps the most interesting question though is the case  $n = 2r + 1$ ; it would be interesting to determine if a stability result is true for any probability  $p$  bounded away from 1. We hope to return to these questions in future work.

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DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK; and DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF MEMPHIS, MEMPHIS TN 38152, USA; and LONDON INSTITUTE FOR MATHEMATICAL SCIENCES, 35A SOUTH ST., MAYFAIR, LONDON W1K 2XF, UK.

*E-mail address:* b.bollobas@dpmms.cam.ac.uk

DEPARTMENT OF PURE MATHEMATICS AND MATHEMATICAL STATISTICS, UNIVERSITY OF CAMBRIDGE, WILBERFORCE ROAD, CAMBRIDGE CB3 0WB, UK

*E-mail address:* b.p.narayanan@dpmms.cam.ac.uk

LOMONOSOV MOSCOW STATE UNIVERSITY, MECHANICS AND MATHEMATICS FACULTY, DEPARTMENT OF MATH. STATISTICS AND RANDOM PROCESSES, LENINSKIE GORY, MOSCOW, 119991, RUSSIA; and MOSCOW INSTITUTE OF PHYSICS AND TECHNOLOGY, FACULTY OF INNOVATIONS AND HIGH TECHNOLOGY, INSTITUTSKIY PER., DOLGOPRUDNY, MOSCOW REGION, 141700, RUSSIA.

*E-mail address:* mraigor@yandex.ru